

Parameterizing interaction of disparate scales: Selective decay by Casimir dissipation in fluids

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Abstract

The problem of parameterizing the interactions of disparate scales in fluid flows is addressed by considering a property of two-dimensional incompressible turbulence. The property we consider is selective decay, in which a Casimir of the ideal formulation (enstrophy in 2D flows) decays in time, while the energy stays essentially constant. This paper introduces a mechanism that produces selective decay by enforcing Casimir dissipation in fluid dynamics. This mechanism turns out to be related in certain cases to the numerical method of anticipated vorticity discussed in [Sadourny and Basdevant \[1981, 1985\]](#). Several examples are given and a general theory of selective decay is developed that uses the Lie-Poisson structure of the ideal theory. A scale-selection operator allows the resulting modifications of the fluid motion equations to be interpreted in several examples as parameterizing the nonlinear, dynamical interactions between disparate scales. The type of modified fluid equation systems derived here may be useful in turbulent geophysical flows where it is computationally prohibitive to rely on the slower, indirect effects of a realistic viscosity, such as in large-scale, coherent, oceanic flows interacting with much smaller eddies.

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1 Introduction

1.1 Symmetries and selective decay

From the viewpoint of Noether's theorem, energy is conserved in ideal fluid dynamics because of the time-translation symmetry of the Lagrangian in Hamilton's principle for ideal fluid motion. A second type of fluid conservation law arises via Noether's theorem because of relabelling symmetry of the Lagrangian. Relabelling symmetries smoothly transform the labels of the fluid parcels without changing the Eulerian quantities on which Hamilton's principle for ideal fluids depends. The conservation laws associated with relabelling symmetries are called Casimirs, because in the Hamiltonian formulation of ideal fluid dynamics in the Eulerian representation their Lie-Poisson brackets with *any* other functionals vanish identically. Thus, the Casimirs arise from a geometric symmetry of the Eulerian representation of ideal fluid dynamics. This relabelling symmetry is also responsible for Kelvin's circulation theorem in ideal fluid dynamics, which immediately leads to the conservation of the Casimirs for 2D ideal incompressible flow, see [Holm, Marsden and Ratiu \[1998\]](#); [Cotter and Holm \[2012\]](#). The Kelvin circulation theorem and the associated Casimir conservation laws are kinematic, because they hold for any choice of Hamiltonian in the Eulerian representation. Energy conservation is dynamic.

The two types of conservation laws for ideal fluids (energy and Casimirs) have been used together previously as the basis for the energy-Casimir method of nonlinear stability analysis of equilibrium flows [HMRW \[1985\]](#). This method is based on the concept of *dynamical balance* in ideal fluids. In particular, the equilibrium solutions of ideal fluid dynamics are characterized by such a balance. Namely, the critical points $\delta(h + \lambda C) = 0$ of any linear combination $h + \lambda C$ of the energy h , a Casimir C and constant λ occur for equilibrium solutions of Euler's equations for ideal fluids. An initial disturbance of this balance may be either stable or unstable in the Lyapunov sense. Sufficient conditions for linear Lyapunov stability of an equilibrium flow may be determined by taking the second variation of the energy-Casimir sum, $\delta^2(h + \lambda C)|_e$, where subscript e means evaluated at the equilibrium flow. This second variation is *conserved* by the linearized equations. This means that linear Lyapunov stability is implied for those flow equilibria for which the second variation $\delta^2(h + \lambda C)|_e$ comprises a norm in the space of perturbations. Thus, in the ideal Hamiltonian description of fluids, the interplay of energy and Casimirs determines equilibrium solutions and governs their Lyapunov stability. For more details, and discussions of the extension of the energy-Casimir method to *nonlinear* Lyapunov stability analysis, see [HMRW \[1985\]](#) and references therein, particularly [Arnold \[1965a,b\]](#) for the case of 2D ideal incompressible fluid flows.

The selective decay hypothesis in 2D turbulence. Already in [Fjortoft \[1953\]](#) it was shown that, in contrast to 3D incompressible flow, the two energy-Casimir constants of motion imply that any transfer of kinetic energy to higher wavenumbers (k) in the 2D incompressible flow of a viscous fluid must be accompanied by a larger transfer of energy to lower wavenumbers. An explanation for

this effect based on statistical mechanics and equilibrium thermodynamics was offered in Kraichnan [1967, 1971], which predicted a forward cascade (k^{-3}) of enstrophy and an inverse cascade ($k^{-5/3}$) of kinetic energy in 2D incompressible turbulence, as reviewed in Kraichnan and Montgomery [1980]. Investigating these predictions numerically, Matthaeus and Montgomery [1980] performed simulations of 2D incompressible turbulence using the Navier-Stokes equations and on this basis postulated the *selective decay* hypothesis. Their selective decay hypothesis was that in the 2D incompressible flow of a viscous fluid the squared L^2 norm of vorticity (or enstrophy, the quadratic Casimir) should decay in time, while the squared L^2 norm of velocity (kinetic energy) which has contributions from both directions in wavenumber should stay essentially constant. The balance between the ($k^{-5/3}$) and (k^{-3}) cascades occurs at the length scale at which energy flux passes through zero, so that energy may be passed in both directions at the same rate simultaneously.

Because the balance induced by the energy-Casimir dynamics is so important in the motion of fluids, one could imagine designing *modified fluid equations* that would govern the balance by *imposing* the selective decay of one or the other of these two types of conservation laws. Moreover, because this balance in the ideal case stems from the Lie-Poisson bracket in the Hamiltonian formulation of ideal fluid dynamics, one could imagine that an effective approach in deriving such modified equations would involve using elements of the Lie-Poisson bracket that would select which type of conservation law would dominate and thus govern the properties of the solution.

The two types of ideal fluid constants of motion, energy and Casimirs, typically have quite different dependences on spatial gradients of the solutions. Consequently, the interplay between them can be interpreted as an interaction between larger and smaller scales (or coherence lengths, or spectral wavenumbers). Thus, imposition of a selective decay mechanism for one or the other type could in principle control the dominance of one set of scales, or sizes of excitations, over another. Hence, modifying the fluid equations to impose selective decay of either the energy or a particular Casimir is interesting, because such a modification could control the direction of the energy cascade, i.e., whether it is forward (toward smaller, finer scales), or backward (toward larger, coarser scales). This means that modifications of the fluid equations that promote selective decay could control the dominant sizes that emerge in the flow and support their stability, by causing smaller perturbations to decay away. Such modifications would control the rate at which the cascade occurs, in either direction, and thus would control the type of solution that emerges from a given initial condition.

Various types of modifications of the Poisson bracket for Hamiltonian systems have previously been proposed that produce energy dissipation. These proposed modifications include: (i) adding a symmetric bilinear form, Kaufman [1984]; Morrison [1984]; Grmela [1984]; Öttinger [2005], and (ii) application of a double bracket, Brockett [1991]; Bloch et al. [1996]; Holm, Putkaradze and Tronci [2008]; Brody, Ellis and Holm [2008]. See also Vallis, Carnevale, and Young [1989], in which a modification of the transport velocity was used to impose energy dissipation with fixed Casimirs.

Here we are not considering energy dissipation. Instead, we are considering a type of selective decay of ideal Casimir constants of motion at fixed energy, as was discussed for 2D turbulence cascades in Matthaeus and Montgomery [1980]. In the present case, the selective decay is enforced by a modification of the vorticity equation that is based on the Lie-Poisson structure of its Hamiltonian formulation in the ideal case Arnold [1966, 1969, 1978]; Holm, Marsden and Ratiu [1998].

Thus, the aim of this paper is to impose selective decay by using the Lie-Poisson structure of the ideal theory, and interpret the resulting modifications of the equations as a means of dynamically

and nonlinearly parameterizing the interactions between disparate scales directly. This type of modification may be useful in situations where it is computationally prohibitive to rely on the slower, indirect effects of viscosity. Moreover, these ideas should apply not only to 2D flow, but also to any 3D flow that is sufficiently anisotropic, particularly in geophysical fluid flows in thin domains.

Plan of the paper. The remainder of this introductory section provides more background about the two types of conservation laws in fluids and defines our selective decay mechanism based on Casimir dissipation in the example of 2D incompressible flows. Remarkably, in that 2D case, the mechanism for selective decay by Casimir dissipation in 2D incompressible fluid dynamics turns out to be related to the numerical method of anticipated vorticity in [Sadourny and Basdevant \[1981, 1985\]](#). Section 2 discusses the general theory of selective decay by Casimir dissipation in the Lie algebraic context that underlies the Lie-Poisson Hamiltonian formulation of ideal fluid dynamics, as explained in, e.g., [Holm, Marsden and Ratiu \[1998\]](#). In particular, we develop the Kelvin circulation theorem and Lagrange-d'Alembert variational principle for Casimir dissipation. In the Lagrange-d'Alembert formulation, the modification of the motion equation to impose selective decay is seen as an energy-conserving constraint force. Section 3 extends the Casimir dissipation theory to include fluids that possess advected quantities such as heat, mass, buoyancy, magnetic field, etc., by using the standard method of Lie-Poisson brackets for semidirect-product actions of Lie groups on vector spaces reviewed in [HMRW \[1985\]](#). Here, the main examples are the rotating shallow water equations and the 3D Boussinesq equations for rotating stratified incompressible fluid flows. Finally, section 4 summarizes our conclusions and suggests some possible next directions for applications of the selective decay modifications of fluid equations, treated as dynamical parameterizations of the interactions between disparate scales.

1.2 2D flows

An interesting feature of ideal incompressible 2D fluid flows is that they have two types of conservation laws which arise from their Hamiltonian formulation in terms of a Lie-Poisson bracket, $\{\cdot, \cdot\}$, [Arnold \[1966, 1969, 1978\]](#), as

$$\frac{df(\omega)}{dt} = \{f, h\}(\omega) = \left\langle \omega, \left[\frac{\delta f}{\delta \omega}, \frac{\delta h}{\delta \omega} \right] \right\rangle := \int_{\mathcal{D}} \omega \left[\frac{\delta f}{\delta \omega}, \frac{\delta h}{\delta \omega} \right] dx dy, \quad (1.1)$$

where ω is the vorticity of the flow, the square bracket $[\cdot, \cdot]$ is the 2D Jacobian, written in various forms as

$$[f, h] = J(f, h) = f_x h_y - h_x f_y = \hat{\mathbf{z}} \cdot \nabla f \times \nabla h = \hat{\mathbf{z}} \cdot \text{curl}(f \hat{\mathbf{z}}) \times \text{curl}(h \hat{\mathbf{z}}), \quad (1.2)$$

and the angle bracket $\langle \cdot, \cdot \rangle$ in (1.1) is the L^2 pairing in the domain \mathcal{D} of the (x, y) plane. For convenience, we shall take the domain \mathcal{D} to be periodic, so we need not worry about boundary terms arising from integrations by parts.

Energy. The first type of conservation law for such 2D ideal fluid flows arises from the antisymmetry of their Lie-Poisson bracket. Namely, antisymmetry of the bracket implies conservation of the Hamiltonian function h

$$\frac{dh(\omega)}{dt} = \{h, h\}(\omega) = 0,$$

for any given choice of h .

Casimirs. The second type of conservation law arises because their Lie-Poisson bracket has a kernel (i.e., is degenerate), which means there exist functions $C(\omega)$ for which

$$\frac{dC(\omega)}{dt} = \{C, h\}(\omega) = 0, \quad (1.3)$$

for any Hamiltonian $h(\omega)$. Functions that satisfy this relation for any Hamiltonian are called *Casimir functions*. (Lie called them distinguished functions, according to Olver [2000].)

Theorem 1.1 (Arnold [1966, 1969, 1978]). The Casimirs for the Lie-Poisson bracket (1.1) in the Hamiltonian formulation of 2D incompressible ideal fluid motion are

$$C_\Phi = \int_{\mathcal{D}} \Phi(\omega) \, dxdy,$$

for any smooth function Φ .

Proof. The well-known proof of this statement follows from an identity for the (x, y) Jacobian bracket $[\cdot, \cdot]$ that arises from integration by parts as

$$\int_{\mathcal{D}} a[b, c] \, dxdy = \int_{\mathcal{D}} b[c, a] \, dxdy = \int_{\mathcal{D}} c[a, b] \, dxdy. \quad (1.4)$$

This identity holds for any smooth functions $a, b, c \in \mathcal{F}(\mathcal{D})$ that vanish on the boundary of the domain \mathcal{D} , and also when the domain is periodic. Thus,

$$\{C_\Phi, h\}(\omega) = \int_{\mathcal{D}} \omega \left[\Phi'(\omega), \frac{\delta h}{\delta \omega} \right] \, dxdy = \int_{\mathcal{D}} \frac{\delta h}{\delta \omega} [\omega, \Phi'(\omega)] \, dxdy = 0,$$

for any Hamiltonian h and function $\Phi(\omega)$. □

Enstrophy. Among the Casimirs $C_\Phi(\omega)$ for 2D ideal fluids is the famous *enstrophy*, given by

$$C_2 := \frac{1}{2} \|\omega\|_{L^2(\mathcal{D})}^2 = \frac{1}{2} \int_{\mathcal{D}} \omega^2 \, dxdy, \quad (1.5)$$

which is the Casimir of the Lie-Poisson bracket (1.1) for the case that $\Phi(\omega) = \frac{1}{2}\omega^2$.

Remark 1.2. Any smooth function of a Casimir (such as its square) is also a Casimir.

Turbulence and selective decay. When viscosity is added, the Navier-Stokes equations for 2D incompressible turbulence result from the equations above. These equations are dissipative and, in the absence of forcing, both types of conserved quantities in the ideal case would decay in time and eventually vanish. Remarkably, in 2D incompressible turbulence these two types of conservation laws are found numerically to decay at *different rates* and this difference is found to have an important emergent effect on the spectral properties and statistics of 2D incompressible turbulence, Kraichnan [1967, 1971]. In particular, this feature of *selective decay* in 2D turbulence

leads to an inverse cascade of energy toward larger scales and a forward cascade of enstrophy toward smaller scales where its dissipation occurs, [Matthaeus and Montgomery \[1980\]](#). This emergent effect of viscosity in 2D incompressible turbulence has had a long history of investigation and is thought to be important in many applications, including climate modelling. Recent work tends to understand this effect as a mechanism for parameterizing the interactions between disparate scales, for example, between large coherent oceanic flows and much smaller eddies, [Marshall and Adcroft \[2010\]](#). For other recent discussions of these ideas in modelling disparate scale interactions in geophysical and astrophysical turbulence, see [Mininni, Pouquet and Sullivan \[2008\]](#).

Question. Given that the two types of conservation laws in ideal 2D flow are both properties of the Lie-Poisson bracket for such flows, and their dissipation has a profound effect of the properties of the flow, it is natural to ask whether one may use the Lie-Poisson bracket of ideal 2D flow to *impose* a process of selective decay.

1.3 Casimir dissipation

The Lie-Poisson bracket may be used to introduce a type of “nonlinear viscosity” that preserves the energy, but dissipates a given Casimir. This may be accomplished naturally by modifying the Hamiltonian formulation to introduce a *quadratic* Lie-Poisson bracket structure, as follows,

$$\frac{df(\omega)}{dt} = \int_{\mathcal{D}} \omega \left[\frac{\delta f}{\delta \omega}, \frac{\delta h}{\delta \omega} \right] dxdy - \theta \int_{\mathcal{D}} \left[\frac{\delta f}{\delta \omega}, \frac{\delta h}{\delta \omega} \right] L \left[\frac{\delta C}{\delta \omega}, \frac{\delta h}{\delta \omega} \right] dxdy, \quad (1.6)$$

where θ is a given constant and L is an arbitrary positive self-adjoint linear differential operator. The differential operator L allows a degree of scale dependence in the Casimir dissipation approach. For example, choosing $L = (1 - \alpha^2 \Delta)^s$ would define a Sobolev $H^s(\mathcal{D})$ inner product with length scale α . One immediately sees from equation (1.6) that

$$\frac{dh(\omega)}{dt} = 0 \quad \text{and} \quad \frac{dC(\omega)}{dt} = -\theta \int_{\mathcal{D}} \left[\frac{\delta C}{\delta \omega}, \frac{\delta h}{\delta \omega} \right] L \left[\frac{\delta C}{\delta \omega}, \frac{\delta h}{\delta \omega} \right] dxdy =: -\theta \left\| \left[\frac{\delta C}{\delta \omega}, \frac{\delta h}{\delta \omega} \right] \right\|_L^2.$$

The latter equation recovers the Hamiltonian case when $\theta = 0$. If a given Casimir $C(\omega)$ is not sign-definite, one may take its square, which is still a Casimir, and find that

$$\frac{1}{2} \frac{dC(\omega)^2}{dt} = -\theta C(\omega)^2 \int_{\mathcal{D}} \left[\frac{\delta C}{\delta \omega}, \frac{\delta h}{\delta \omega} \right] L \left[\frac{\delta C}{\delta \omega}, \frac{\delta h}{\delta \omega} \right] dxdy = -\theta C(\omega)^2 \left\| \left[\frac{\delta C}{\delta \omega}, \frac{\delta h}{\delta \omega} \right] \right\|_L^2 \quad (1.7)$$

for ω verifying the equations (1.6) associated to the Casimir function $\frac{1}{2}C(\omega)^2$. These calculations have demonstrated the following.

Proposition 1.3. For $\theta > 0$ in equation (1.6), the energy Hamiltonian $h(\omega)$ is conserved and squared Casimirs $C(\omega)^2$ decay exponentially at a rate proportional to the square of the (possibly degenerate) norm induced by the L -pairing with linear operator L .

Remark 1.4. L is called a *scale-selective* operator, because choosing it to contain higher spatial derivatives emphasizes the higher wavenumbers in the L -norm in equation (1.7). This, in turn, determines the Casimir dissipation rate. Thus, the choice of the operator L establishes the range

of sizes of spatial scales which most contribute to the modifications of the equations due to Casimir dissipation. In particular, if one were to take

$$L = (1 - \alpha^2 \Delta)^s, \quad \text{so that} \quad \left\| \left[\frac{\delta C}{\delta \omega}, \frac{\delta h}{\delta \omega} \right] \right\|_L^2 = \left\| \left[\frac{\delta C}{\delta \omega}, \frac{\delta h}{\delta \omega} \right] \right\|_{H^s}^2,$$

then scale sizes greater or less than the length α would tend to behave differently.

Double Lie-Poisson bracket vorticity dynamics (1.6) in 2D. The 2D vorticity dynamics generated by the quadratic Lie-Poisson bracket structure in equation (1.6) is found by choosing $f(\omega) = \omega$ above, for which

$$\begin{aligned} \frac{\partial \omega}{\partial t} &= \left[\frac{\delta h}{\delta \omega}, \omega \right] + \theta \left[\frac{\delta h}{\delta \omega}, L \left[\frac{\delta h}{\delta \omega}, \frac{\delta C}{\delta \omega} \right] \right] \\ &= \left[\frac{\delta h}{\delta \omega}, \omega + \theta L \left[\frac{\delta h}{\delta \omega}, \frac{\delta C}{\delta \omega} \right] \right], \end{aligned} \tag{1.8}$$

after using the identity in (1.4) obtained from integration by parts.

Remark 1.5 (Double bracket dissipation structure and “anticipated vorticity”).

- When the operator $L = \text{Id}$, equation (1.8) defines a *double* bracket dissipation structure whose formula is reminiscent of the double bracket dissipation introduced in Bloch et al. [1996] and Holm, Putkaradze and Tronci [2008] which would yield here the dissipation term $\theta [\omega, [\omega, \frac{\delta h}{\delta \omega}]]$. Both the similarities and differences between these two approaches can be clearly seen from their respective abstract Lie algebraic formulations, see Remark 2.6 later.
- When one sets $\delta h / \delta \omega = \psi$, with stream function $\psi(x, y)$, and chooses the Casimir to be the enstrophy $C = \frac{1}{2} \int_{\mathcal{D}} \omega^2 dx dy$, equation (1.8) recovers the *anticipated vorticity model* (AVM) of Sadourny and Basdevant [1981, 1985] for an appropriate choice of L . See also Vallis and Hua [1988]. Namely, in terms of the stream function representation of the 2D vector velocity $\mathbf{u} = \text{curl}(\psi \hat{\mathbf{z}}) = -\hat{\mathbf{z}} \times \nabla \psi$ with vorticity $\omega \hat{\mathbf{z}} = \text{curl curl}(\psi \hat{\mathbf{z}}) = -\Delta \psi \hat{\mathbf{z}}$, equation (1.8) above with $L = \text{Id}$ and $\delta h / \delta \omega = \psi$ becomes

$$\begin{aligned} \frac{\partial \omega}{\partial t} &= [\psi, \omega] + \theta [\psi, L[\psi, \omega]] = [\psi, \omega + \theta L[\psi, \omega]] \\ &= -\mathbf{u} \cdot \nabla \omega + \theta \mathbf{u} \cdot \nabla L(\mathbf{u} \cdot \nabla \omega) = -\mathbf{u} \cdot \nabla (\omega - \theta L(\mathbf{u} \cdot \nabla \omega)). \end{aligned} \tag{1.9}$$

The word “anticipated” was used in Sadourny and Basdevant [1981, 1985] when naming AVM, because if one were to take $\theta \simeq \Delta t$, where Δt is the time step of the numerical method, the term proportional to θ in the last parenthesis of (1.9) would approximate the vorticity at $t + \Delta t$ for the pure Hamiltonian vorticity evolution at linear order in Δt . The AVM approach has been very intriguing in the modelling of 2D incompressible turbulence and much has been written about it in that literature. These ideas and particularly the choice of the time-scale parameter θ have also been discussed recently in the context of shallow water modelling, see, e.g., Chen, Gunzburger, and Ringler [2011a,b]. The other *scale-selective* AVM models considered in Sadourny and Basdevant [1981, 1985] can be obtained by making an appropriate choice for the differential operator L in equation (1.6), or (1.8). Sadourny and Basdevant [1981, 1985] take $L = \Delta^8$, corresponding to the Sobolev space H^8 .

Remark 1.6 (Quasigeostrophy). The flexibility of the Casimir dissipation approach can be shown by passing to the case of 2D rotating quasigeostrophic fluids (QG), for which the vorticity ω is replaced by *potential* vorticity q satisfying the relation $\delta h / \delta q = \psi$ and equation (1.9) becomes

$$\frac{\partial q}{\partial t} = -\mathbf{u} \cdot \nabla (q - \theta L(\mathbf{u} \cdot \nabla q)), \quad \mathbf{u} = -\hat{\mathbf{z}} \times \nabla \psi, \quad q = -\Delta \psi + \mathcal{F} \psi + f,$$

where the constants \mathcal{F} and f denote the square of the inverse Rossby radius and the rotation frequency, respectively.

Double Lie-Poisson bracket vorticity dynamics in 3D. Our goal now is to write the quadratic Lie-Poisson bracket structure (1.6) in any number of dimensions and illustrate its effects in several hopefully illuminating examples. We begin by using (1.2) in 2D to extend the quadratic vorticity bracket expression (1.6) from 2D to 3D, as follows,

$$\begin{aligned} \frac{df(\boldsymbol{\omega})}{dt} = \{f, h\}(\boldsymbol{\omega}) &= \int_{\mathcal{D}} \boldsymbol{\omega} \cdot \text{curl} \frac{\delta f}{\delta \boldsymbol{\omega}} \times \text{curl} \frac{\delta h}{\delta \boldsymbol{\omega}} d^3x \\ &\quad - \theta \int_{\mathcal{D}} \text{curl} \frac{\delta f}{\delta \boldsymbol{\omega}} \times \text{curl} \frac{\delta h}{\delta \boldsymbol{\omega}} \cdot L \left(\text{curl} \frac{\delta C}{\delta \boldsymbol{\omega}} \times \text{curl} \frac{\delta h}{\delta \boldsymbol{\omega}} \right) d^3x. \end{aligned} \quad (1.10)$$

The first (skewsymmetric, Lie-Poisson) summand in this modified vorticity bracket may be written in terms of velocity equivalently as

$$\begin{aligned} \{f, h\}_{1st}(\mathbf{u}) &= \int_{\mathcal{D}} \boldsymbol{\omega} \cdot \text{curl} \frac{\delta f}{\delta \boldsymbol{\omega}} \times \text{curl} \frac{\delta h}{\delta \boldsymbol{\omega}} d^3x \\ &= \int_{\mathcal{D}} \mathbf{u} \cdot \text{curl} \left(\frac{\delta f}{\delta \mathbf{u}} \times \frac{\delta h}{\delta \mathbf{u}} \right) d^3x \\ &= - \int_{\mathcal{D}} \mathbf{u} \cdot \left[\frac{\delta f}{\delta \mathbf{u}}, \frac{\delta h}{\delta \mathbf{u}} \right] d^3x, \end{aligned}$$

where now in 3D the square brackets $[\cdot, \cdot]$ represent Lie brackets of divergence-free vector fields, and we have used the identity $[\mathbf{u}, \mathbf{v}] = -\text{curl}(\mathbf{u} \times \mathbf{v})$ for $\nabla \cdot \mathbf{u} = 0 = \nabla \cdot \mathbf{v}$.

The second (inner product) summand in the vorticity bracket (1.10) depends on the Casimir C and may be written in terms of velocity equivalently as

$$\begin{aligned} \{f, h\}_{2nd}(\mathbf{u}) &= -\theta \int_{\mathcal{D}} \text{curl} \frac{\delta f}{\delta \boldsymbol{\omega}} \times \text{curl} \frac{\delta h}{\delta \boldsymbol{\omega}} \cdot L \left(\text{curl} \frac{\delta C}{\delta \boldsymbol{\omega}} \times \text{curl} \frac{\delta h}{\delta \boldsymbol{\omega}} \right) d^3x \\ &= -\theta \int_{\mathcal{D}} \left(\frac{\delta f}{\delta \mathbf{u}} \times \frac{\delta h}{\delta \mathbf{u}} \right) \cdot L \left(\frac{\delta C}{\delta \mathbf{u}} \times \frac{\delta h}{\delta \mathbf{u}} \right) d^3x \\ &= -\theta \int_{\mathcal{D}} \text{curl} \left(\frac{\delta f}{\delta \mathbf{u}} \times \frac{\delta h}{\delta \mathbf{u}} \right) \cdot \text{curl}^{-1} L \text{curl}^{-1} \text{curl} \left(\frac{\delta C}{\delta \mathbf{u}} \times \frac{\delta h}{\delta \mathbf{u}} \right) d^3x \\ &= -\theta \int_{\mathcal{D}} \left[\frac{\delta f}{\delta \mathbf{u}}, \frac{\delta h}{\delta \mathbf{u}} \right] \cdot \text{curl}^{-1} L \text{curl}^{-1} \left[\frac{\delta C}{\delta \mathbf{u}}, \frac{\delta h}{\delta \mathbf{u}} \right] d^3x \\ &=: -\theta \gamma \left(\left[\frac{\delta f}{\delta \mathbf{u}}, \frac{\delta h}{\delta \mathbf{u}} \right], \left[\frac{\delta C}{\delta \mathbf{u}}, \frac{\delta h}{\delta \mathbf{u}} \right] \right). \end{aligned}$$

Here, the positive operator $\Lambda := \operatorname{curl}^{-1} L \operatorname{curl}^{-1}$ defines a symmetric bilinear form γ on velocities. That is, on the Lie algebra of divergence-free vector fields we have $\gamma : \mathfrak{X} \times \mathfrak{X} \rightarrow \mathbb{R}$

$$\gamma(\mathbf{u}, \mathbf{v}) := \int_{\mathcal{D}} \mathbf{u} \cdot \Lambda \mathbf{v} \, d^3x := \langle \mathbf{u}, \mathbf{v}^b \rangle .$$

The second equality here defines a pairing $\langle \cdot, \cdot \rangle : \mathfrak{X} \times \mathfrak{X}^* \rightarrow \mathbb{R}$, with the flat-operator $\flat : \mathfrak{X} \rightarrow \mathfrak{X}^*$ given by $\mathbf{v}^b = (\Lambda \mathbf{v}) \cdot d\mathbf{x}$.

Having investigated the structure of the modified vorticity bracket for vorticity $\boldsymbol{\omega}$ in equation (1.10), we may now write the modified equation for the vorticity, as

$$\begin{aligned} \frac{\partial \boldsymbol{\omega}}{\partial t} &= \operatorname{curl} \left(\operatorname{curl} \frac{\delta h}{\delta \boldsymbol{\omega}} \times \boldsymbol{\omega} \right) - \theta \operatorname{curl} \left(\operatorname{curl} \frac{\delta h}{\delta \boldsymbol{\omega}} \times L \left(\operatorname{curl} \frac{\delta C}{\delta \boldsymbol{\omega}} \times \operatorname{curl} \frac{\delta h}{\delta \boldsymbol{\omega}} \right) \right) \\ &= \operatorname{curl} \left[\operatorname{curl} \frac{\delta h}{\delta \boldsymbol{\omega}} \times \left(\boldsymbol{\omega} - \theta L \left(\operatorname{curl} \frac{\delta C}{\delta \boldsymbol{\omega}} \times \operatorname{curl} \frac{\delta h}{\delta \boldsymbol{\omega}} \right) \right) \right] . \end{aligned} \quad (1.11)$$

Conserved quantities in 3D. In 3D the fluid kinetic energy is expressed in terms of its vorticity as

$$h(\boldsymbol{\omega}) = \frac{1}{2} \int_{\mathcal{D}} |\mathbf{u}|^2 \, d^3x = \frac{1}{2} \int_{\mathcal{D}} \boldsymbol{\omega} \cdot (-\Delta^{-1} \boldsymbol{\omega}) \, d^3x \quad (1.12)$$

and the only Casimir in the Lie-Poisson Hamiltonian formulation is the helicity

$$C(\boldsymbol{\omega}) = \frac{1}{2} \int_{\mathcal{D}} \boldsymbol{\omega} \cdot \mathbf{u} \, d^3x = \frac{1}{2} \int_{\mathcal{D}} \boldsymbol{\omega} \cdot \operatorname{curl}^{-1} \boldsymbol{\omega} \, d^3x , \quad (1.13)$$

in which we may take $\operatorname{curl}^{-1} \boldsymbol{\omega} = \mathbf{u}$ to be divergence-free. The helicity represents the total linkage of the lines of vorticity with itself. It is preserved by the Euler fluid equations, but is dissipated by the viscous Navier-Stokes equations. For more details and discussions of the nature of fluid helicity, see [Arnold and Khesin \[1998\]](#). For the choices of $h(\boldsymbol{\omega})$ and $C(\boldsymbol{\omega})$ above, we have

$$\frac{\delta h}{\delta \mathbf{u}} = \operatorname{curl} \frac{\delta h}{\delta \boldsymbol{\omega}} = \mathbf{u} \quad \text{and} \quad \frac{\delta C}{\delta \mathbf{u}} = \operatorname{curl} \frac{\delta C}{\delta \boldsymbol{\omega}} = \operatorname{curl} \mathbf{u} = \boldsymbol{\omega} . \quad (1.14)$$

Consequently, the vorticity dynamics equation (1.11) for these choices becomes

$$\begin{aligned} \frac{\partial \boldsymbol{\omega}}{\partial t} &= \operatorname{curl} (\mathbf{u} \times (\boldsymbol{\omega} - \theta L(\boldsymbol{\omega} \times \mathbf{u}))) \\ &= - \left[\mathbf{u}, \boldsymbol{\omega} - \theta \operatorname{curl} (\Lambda [\mathbf{u}, \boldsymbol{\omega}]) \right] , \end{aligned} \quad (1.15)$$

and when the operator $L = \operatorname{Id}$, we have $\Lambda = -\Delta^{-1}$ (the inverse Laplacian) and $\operatorname{curl} \Lambda = \operatorname{curl}^{-1}$. In velocity form, these equations read

$$\partial_t \mathbf{u} + (\boldsymbol{\omega} - \theta L(\boldsymbol{\omega} \times \mathbf{u})) \times \mathbf{u} = -\nabla p \quad (1.16)$$

Remark 1.7 (Stationary solutions and generalized Beltrami flows and Lamb surfaces).

Stationary solutions of (1.16) satisfy

$$(\boldsymbol{\omega} - \theta L(\boldsymbol{\omega} \times \mathbf{u})) \times \mathbf{u} = -\nabla p .$$

This stationary flow condition means that either the vectors of velocity \mathbf{u} and modified vorticity $\boldsymbol{\omega} - \theta L(\boldsymbol{\omega} \times \mathbf{u})$ are parallel, a subcase of which is $\boldsymbol{\omega} \times \mathbf{u} = 0$ (Beltrami flows), or that the vectors \mathbf{u} and $\boldsymbol{\omega} - \theta L(\boldsymbol{\omega} \times \mathbf{u})$ are both tangent to the same pressure surface $p(\mathbf{x}) = \text{const}$, *viz*

$$\mathbf{u} \cdot \nabla p = 0 \quad \text{and} \quad (\boldsymbol{\omega} - \theta L(\boldsymbol{\omega} \times \mathbf{u})) \cdot \nabla p = 0. \quad (1.17)$$

For $\theta = 0$, a surface defined by $p(\mathbf{x}) = \text{const}$ for steady flows would be called a Lamb surface, see, e.g., [Holm \[2011\]](#); so surfaces satisfying (1.17) may be called *generalized Lamb surfaces*. The geometry of generalized Beltrami flows and Lamb surfaces may be an interesting research topic.

Remark 1.8. Equation (1.15) is the 3D version of the anticipated vorticity equation (1.9) in 2D with L replaced by $\text{curl} \Lambda = L \text{curl}^{-1}$. Again, the constant $\theta \simeq \Delta t$ has units of time. A “scale-aware” version of this anticipated vorticity method in 3D may be obtained by selecting the value of the time-scale parameter θ based on local mean properties of the solution, instead of the time step of the numerical method. However, then the issue arises of whether the time-scale parameter θ should be part of the symmetric operator L and, thus, whether its spatial dependence should play a role. This is beyond the scope of the present work. For a discussion of a numerical algorithm with scale awareness for 2D multiresolution grids, see [Chen, Gunzburger, and Ringler \[2011b\]](#).

Anticipated Kelvin circulation. Inverting the curl in equation (1.15) yields

$$\begin{aligned} (\partial_t + \mathcal{L}_u)(\mathbf{u} \cdot d\mathbf{x}) &= -dp + \theta \mathcal{L}_u (\text{curl}^{-1} L(\boldsymbol{\omega} \times \mathbf{u}) \cdot d\mathbf{x}) \\ &= -dp + \theta \mathcal{L}_u (\text{curl}^{-1} L \text{curl}^{-1}(\text{curl}(\boldsymbol{\omega} \times \mathbf{u})) \cdot d\mathbf{x}) \\ &= -dp + \theta \mathcal{L}_u (\Lambda[\mathbf{u}, \boldsymbol{\omega}] \cdot d\mathbf{x}), \end{aligned} \quad (1.18)$$

where, we have substituted $\Lambda := \text{curl}^{-1} L \text{curl}^{-1}$ and used $[\mathbf{u}, \boldsymbol{\omega}] = -\text{curl}(\mathbf{u} \times \boldsymbol{\omega})$. The symbol \mathcal{L}_u denotes the Lie derivative with respect to the velocity vector field $u = \mathbf{u} \cdot \nabla$, expressible when applied to 1-forms in 3D as

$$\mathcal{L}_u(\mathbf{v} \cdot d\mathbf{x}) = -(\mathbf{u} \times \text{curl} \mathbf{v}) \cdot d\mathbf{x} + d(\mathbf{u} \cdot \mathbf{v}). \quad (1.19)$$

For an introduction to the use of Lie derivatives in fluid mechanics, see [Holm \[2011\]](#). For a more advanced discussion, see [Holm, Marsden and Ratiu \[1998\]](#).

Kevin’s circulation theorem now becomes, cf. equation (1.15)

$$\begin{aligned} \frac{d}{dt} \oint_{c(\mathbf{u})} \mathbf{u} \cdot d\mathbf{x} &= \oint_{c(\mathbf{u})} (\partial_t + \mathcal{L}_u)(\mathbf{u} \cdot d\mathbf{x}) = -\theta \oint_{c(\mathbf{u})} \mathbf{u} \times \text{curl} (\Lambda[\mathbf{u}, \boldsymbol{\omega}]) \cdot d\mathbf{x} \\ &= -\theta \oint_{c(\mathbf{u})} \mathbf{u} \times L(\boldsymbol{\omega} \times \mathbf{u}) \cdot d\mathbf{x}. \end{aligned} \quad (1.20)$$

The last term is the source of circulation due to Casimir dissipation.

Variational-derivative expressions. Recall that, in terms of the velocity, we have the variational expressions

$$\frac{\delta h}{\delta \mathbf{u}} = \mathbf{u} \quad \text{and} \quad \frac{\delta C}{\delta \mathbf{u}} = \boldsymbol{\omega} \quad (C = \text{helicity}),$$

so the last line of the calculation in (1.18) may be expressed in terms of variational derivatives as

$$(\partial_t + \mathcal{L}_{\delta h / \delta \mathbf{u}}) (\mathbf{u} \cdot d\mathbf{x}) = -dp + \theta \mathcal{L}_{\delta h / \delta \mathbf{u}} \left[\frac{\delta h}{\delta \mathbf{u}}, \frac{\delta C}{\delta \mathbf{u}} \right]^\flat, \quad (1.21)$$

This type of expression in terms of variational derivatives will also be obtained by using the general theory developed in the next section.

Proposition 1.9 (Squared Casimir dissipation – Helicity). Let the 3D version of equation (1.7) for squared Casimir dissipation be applied to the square of the helicity, with

$$C(\boldsymbol{\omega}) = \frac{1}{2} \int_{\mathcal{D}} \boldsymbol{\omega} \cdot \text{curl}^{-1} \boldsymbol{\omega} \, d^3x \quad \text{and} \quad h(\boldsymbol{\omega}) = \frac{1}{2} \int_{\mathcal{D}} |\text{curl}^{-1} \boldsymbol{\omega}|^2 \, d^3x = \frac{1}{2} \int_{\mathcal{D}} |\mathbf{u}|^2 \, d^3x.$$

Then the squared helicity Casimir dissipates according to the equation

$$\begin{aligned} \frac{1}{2} \frac{d C(\boldsymbol{\omega})^2}{dt} &= -\theta C(\boldsymbol{\omega})^2 \gamma([\boldsymbol{\omega}, \mathbf{u}], [\boldsymbol{\omega}, \mathbf{u}]) \\ &=: -\theta C(\boldsymbol{\omega})^2 \|\llbracket \mathbf{u}, \boldsymbol{\omega} \rrbracket\|_{\Lambda}^2 = -\theta C(\boldsymbol{\omega})^2 \|\mathbf{u} \times \boldsymbol{\omega}\|_L^2. \end{aligned} \quad (1.22)$$

Proof. The proof is a direct calculation, starting from equation (1.10).

$$\begin{aligned} \frac{1}{2} \frac{d C(\boldsymbol{\omega})^2}{dt} &= -\theta C(\boldsymbol{\omega})^2 \int_{\mathcal{D}} \text{curl} \frac{\delta C}{\delta \boldsymbol{\omega}} \times \text{curl} \frac{\delta h}{\delta \boldsymbol{\omega}} \cdot L \left(\text{curl} \frac{\delta C}{\delta \boldsymbol{\omega}} \times \text{curl} \frac{\delta h}{\delta \boldsymbol{\omega}} \right) d^3x \\ &= -\theta C(\boldsymbol{\omega})^2 \int_{\mathcal{D}} \text{curl}^{-1} \left[\text{curl} \frac{\delta C}{\delta \boldsymbol{\omega}}, \text{curl} \frac{\delta h}{\delta \boldsymbol{\omega}} \right] \cdot L \text{curl}^{-1} \left[\text{curl} \frac{\delta C}{\delta \boldsymbol{\omega}}, \text{curl} \frac{\delta h}{\delta \boldsymbol{\omega}} \right] d^3x \\ &= -\theta C(\boldsymbol{\omega})^2 \int_{\mathcal{D}} \left[\text{curl} \frac{\delta C}{\delta \boldsymbol{\omega}}, \text{curl} \frac{\delta h}{\delta \boldsymbol{\omega}} \right] \cdot \Lambda \left[\text{curl} \frac{\delta C}{\delta \boldsymbol{\omega}}, \text{curl} \frac{\delta h}{\delta \boldsymbol{\omega}} \right] d^3x \\ &= -\theta C(\boldsymbol{\omega})^2 \int_{\mathcal{D}} [\boldsymbol{\omega}, \mathbf{u}] \cdot \Lambda [\boldsymbol{\omega}, \mathbf{u}] \, d^3x \quad \text{with} \quad \Lambda := \text{curl}^{-1} L \text{curl}^{-1} \\ &= -\theta C(\boldsymbol{\omega})^2 \gamma([\boldsymbol{\omega}, \mathbf{u}], [\boldsymbol{\omega}, \mathbf{u}]) \\ &=: -\theta C(\boldsymbol{\omega})^2 \|\llbracket \boldsymbol{\omega}, \mathbf{u} \rrbracket\|_{\Lambda}^2. \end{aligned} \quad (1.23)$$

A shorter proof may be obtained by replacing the functional derivatives directly in the first line to get the equivalent result,

$$\frac{1}{2} \frac{d C(\boldsymbol{\omega})^2}{dt} = -\theta C(\boldsymbol{\omega})^2 \|\boldsymbol{\omega} \times \mathbf{u}\|_L^2,$$

and then substituting the interesting identity that relates the L - and Λ -norms,

$$\|\boldsymbol{\omega} \times \mathbf{u}\|_L^2 = \|\llbracket \boldsymbol{\omega}, \mathbf{u} \rrbracket\|_{\Lambda}^2. \quad (1.24)$$

□

2 Selective decay by Casimir dissipation: General theory

It is clear that the modification process in the previous section produces the anticipated equations in the cases discussed so far, but despite appearances in those examples, this Casimir dissipation process involves considerably more than merely iterating the evolution operator.

2.1 The modified Lie-Poisson (LP) framework

To describe the modified Lie-Poisson structure for the general theory, we fix a Lie algebra \mathfrak{g} with Lie brackets denoted by $[\cdot, \cdot]$, and consider a space \mathfrak{g}^* in weak nondegenerate duality with \mathfrak{g} , that is, there exists a pairing $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R}$, such that for any $\xi \in \mathfrak{g}$, the condition $\langle \mu, \xi \rangle = 0$, for all $\mu \in \mathfrak{g}^*$ implies $\xi = 0$ and, similarly, for any $\mu \in \mathfrak{g}^*$, the condition $\langle \mu, \xi \rangle = 0$ for all $\xi \in \mathfrak{g}$ implies $\mu = 0$. Recall that \mathfrak{g}^* carries a natural Poisson structure, called the Lie-Poisson structure, given by

$$\{f, h\}_+(\mu) = \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta h}{\delta \mu} \right] \right\rangle, \quad (2.1)$$

(Marsden and Ratiu [1994]) where $f, g \in \mathcal{F}(\mathfrak{g}^*)$ are real valued functions defined on \mathfrak{g}^* and $\delta f / \delta \mu \in \mathfrak{g}$ denotes the functional derivative, defined through the duality pairing $\langle \cdot, \cdot \rangle$, by

$$\left\langle \frac{\delta f}{\delta \mu}, \delta \mu \right\rangle = \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} f(\mu + \varepsilon \delta \mu).$$

This Poisson bracket is obtained by reduction of the canonical Poisson structure on the phase space T^*G of the Lie group G with Lie algebra \mathfrak{g} . The symmetry underlying this reduction is given by right translation by G on T^*G . In the case of ideal fluid motion, this symmetry corresponds to relabelling symmetry of the Lagrangian in Hamilton's principle.

Recall that a function $C : \mathfrak{g}^* \rightarrow \mathbb{R}$ is a *Casimir function* for the Lie-Poisson structure (2.1) if it verifies $\{C, f\}_+ = 0$ for all functions $f \in \mathcal{F}(\mathfrak{g}^*)$ or, equivalently $\text{ad}_{\frac{\delta C}{\delta \mu}}^* \mu = 0$, for all $\mu \in \mathfrak{g}^*$, where $\text{ad}_{\xi}^* : \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ is the coadjoint operator defined by $\langle \text{ad}_{\xi}^* \mu, \eta \rangle = \langle \mu, [\xi, \eta] \rangle$.

Below, we will denote by γ_μ be a (possibly μ -dependent) symmetric bilinear form $\gamma_\mu : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$. This form is said to be *positive* if

$$\gamma_\mu(\xi, \xi) \geq 0, \quad \text{for all } \xi \in \mathfrak{g}.$$

Definition 2.1. *Given a Casimir function $C(\mu)$, a positive symmetric bilinear form γ_μ , and a real number $\theta > 0$, we consider the following modification of the LP (Lie-Poisson) equation to produce the Casimir dissipative LP equation:*

$$\partial_t \mu + \text{ad}_{\frac{\delta h}{\delta \mu}}^* \mu = \theta \text{ad}_{\frac{\delta h}{\delta \mu}}^* \left[\frac{\delta C}{\delta \mu}, \frac{\delta h}{\delta \mu} \right]^\flat, \quad (2.2)$$

where $\flat : \mathfrak{g} \rightarrow \mathfrak{g}^*$ is the flat operator associated to γ_μ , that is, the linear form $\xi^\flat \in \mathfrak{g}^*$ is given by $\xi^\flat(\eta) = \gamma_\mu(\xi, \eta)$, for all $\xi, \eta \in \mathfrak{g}$.

Note that the flat operator need not be either injective or surjective. Note also that in the equations (2.2) above, the flat operator is evaluated at μ . It is important to observe that the modification term depends on both the given Hamiltonian function h and the chosen Casimir C . It is convenient to write (2.2) as

$$\partial_t \mu + \text{ad}_{\frac{\delta h}{\delta \mu}}^* \tilde{\mu} = 0, \quad (2.3)$$

for the modified momentum $\tilde{\mu} := \mu + \theta \left[\frac{\delta h}{\delta \mu}, \frac{\delta C}{\delta \mu} \right]^\flat$.

Remark 2.2 (Left-invariant case). Recall that the Lie-Poisson structure (2.1) is associated to *right* G -invariance on T^*G . We have made this choice because ideal fluids are naturally right-invariant systems in the Eulerian representation. Other systems, such as rigid bodies, are *left* G -invariant. In this case, one obtains the Lie-Poisson brackets $\{f, g\}_-(\mu) = -\left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta h}{\delta \mu} \right] \right\rangle$ and this leads to the following change of sign in the Casimir dissipative LP equation (2.2):

$$\partial_t \mu - \text{ad}_{\frac{\delta h}{\delta \mu}}^* \mu = \theta \text{ad}_{\frac{\delta h}{\delta \mu}}^* \left[\frac{\delta C}{\delta \mu}, \frac{\delta h}{\delta \mu} \right]^b. \quad (2.4)$$

The rigid body example. For the rigid body case itself, one identifies μ in (2.4) with $\mathbf{\Pi} \in \mathbb{R}^3$, the body angular momentum, and $\delta h / \delta \mu$ with $\mathbf{\Omega} \in \mathbb{R}^3$, the body angular velocity, while the Casimir is $C = \frac{1}{2} |\mathbf{\Pi}|^2$. Then, choosing for γ the usual inner product on \mathbb{R}^3 , (2.4) becomes

$$\frac{d\mathbf{\Pi}}{dt} + \mathbf{\Omega} \times \mathbf{\Pi} = -\theta \mathbf{\Omega} \times (\mathbf{\Pi} \times \mathbf{\Omega}). \quad (2.5)$$

Hence, the Casimir is dissipated by this modified rigid body equation when $\theta > 0$, since

$$\frac{d}{dt} \frac{1}{2} |\mathbf{\Pi}|^2 = -\theta \mathbf{\Pi} \cdot \mathbf{\Omega} \times (\mathbf{\Pi} \times \mathbf{\Omega}) = -\theta |\mathbf{\Omega} \times \mathbf{\Pi}|^2. \quad (2.6)$$

Remark 2.3 (Casimir dissipation VS anticipated setting). Clearly, equation (2.2) is not equivalent to a simple iteration of the evolution operator, as might be considered in an “anticipated” setting in the form

$$\partial_t \mu + \text{ad}_{\frac{\delta h}{\delta \mu}}^* \mu = \theta \text{ad}_{\frac{\delta h}{\delta \mu}}^* \left(\text{ad}_{\frac{\delta h}{\delta \mu}}^* \mu \right). \quad (2.7)$$

However, the “anticipated” modification does not guarantee selective decay of Casimirs, $dC/dt < 0$, in general. The dynamics of equations (2.2) for Casimir dissipation and (2.7) for “anticipated” motion are only equivalent when

$$\text{ad}_{\frac{\delta h}{\delta \mu}}^* \left[\frac{\delta C}{\delta \mu}, \frac{\delta h}{\delta \mu} \right]^b = \text{ad}_{\frac{\delta h}{\delta \mu}}^* \left(\text{ad}_{\frac{\delta h}{\delta \mu}}^* \mu \right), \quad (2.8)$$

which can only hold for quadratic Casimirs C . Thus, “anticipated” motion and selective Casimir decay can only coincide for certain quadratic Casimirs.

Lie-Poisson formulation. In Lie-Poisson form, the Casimir dissipation equation (2.2) reads

$$\begin{aligned} \frac{df(\mu)}{dt} &= \left\langle \frac{\delta f}{\delta \mu}, \partial_t \mu \right\rangle = - \left\langle \frac{\delta f}{\delta \mu}, \text{ad}_{\frac{\delta h}{\delta \mu}}^* \mu \right\rangle + \theta \left\langle \frac{\delta f}{\delta \mu}, \text{ad}_{\frac{\delta h}{\delta \mu}}^* \left[\frac{\delta C}{\delta \mu}, \frac{\delta h}{\delta \mu} \right]^b \right\rangle \\ &= \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta h}{\delta \mu} \right] \right\rangle - \theta \left\langle \left[\frac{\delta f}{\delta \mu}, \frac{\delta h}{\delta \mu} \right], \left[\frac{\delta C}{\delta \mu}, \frac{\delta h}{\delta \mu} \right]^b \right\rangle \\ &= \{f, h\}_+ - \theta \gamma \left(\left[\frac{\delta f}{\delta \mu}, \frac{\delta h}{\delta \mu} \right], \left[\frac{\delta C}{\delta \mu}, \frac{\delta h}{\delta \mu} \right] \right), \end{aligned} \quad (2.9)$$

for an arbitrary function $f : \mathfrak{g}^* \rightarrow \mathbb{R}$. The energy is preserved, since we have

$$\frac{dh(\mu)}{dt} = \{h, h\}_+ - \theta \gamma \left(\left[\frac{\delta h}{\delta \mu}, \frac{\delta h}{\delta \mu} \right], \left[\frac{\delta C}{\delta \mu}, \frac{\delta h}{\delta \mu} \right] \right) = 0.$$

However, when $\theta > 0$ the Casimir function C is dissipated since

$$\frac{dC(\mu)}{dt} = \{C, h\}_+ - \theta \gamma \left(\left[\frac{\delta C}{\delta \mu}, \frac{\delta h}{\delta \mu} \right], \left[\frac{\delta C}{\delta \mu}, \frac{\delta h}{\delta \mu} \right] \right) = -\theta \left\| \left[\frac{\delta C}{\delta \mu}, \frac{\delta h}{\delta \mu} \right] \right\|_\gamma^2, \quad (2.10)$$

where $\|\xi\|_\gamma^2 := \gamma_\mu(\xi, \xi)$ is the quadratic form (possibly degenerate) associated to the positive bilinear form γ_μ .

Left invariant case. In the left invariant case, we have, in comparison with (2.9),

$$\frac{df(\mu)}{dt} = \{f, h\}_- - \theta \gamma \left(\left[\frac{\delta f}{\delta \mu}, \frac{\delta h}{\delta \mu} \right], \left[\frac{\delta C}{\delta \mu}, \frac{\delta h}{\delta \mu} \right] \right).$$

Kelvin-Noether theorem. The well-known Kelvin circulation theorems for fluids can be seen as reformulations of Noether's theorem and, therefore, they have an abstract Lie algebraic formulation (the Kelvin-Noether theorems), see [Holm, Marsden and Ratiu \[1998\]](#). We now examine how the dissipation term introduced above modifies the abstract Kelvin circulation theorem.

In order to formulate the Kelvin-Noether theorem, one has to choose a manifold \mathcal{C} on which the group G acts on the left and consider a G -equivariant map $\mathcal{K} : \mathcal{C} \rightarrow \mathfrak{g}^{**}$, i.e. $\langle \mathcal{K}(gc), \text{Ad}_{g^{-1}}^* \nu \rangle = \langle \mathcal{K}(c), \nu \rangle, \forall g \in G$. Here gc denotes the action of $g \in G$ on $c \in \mathcal{C}$ and Ad_g^* denotes the coadjoint action defined by $\langle \text{Ad}_g^* \mu, \xi \rangle = \langle \mu, \text{Ad}_g \xi \rangle$, where $\mu \in \mathfrak{g}^*$, $\xi \in \mathfrak{g}$, and Ad_g is the adjoint action of G on \mathfrak{g} . Given $c \in \mathcal{C}$ and $\mu \in \mathfrak{g}^*$, we will refer to $\langle \mathcal{K}(c), \mu \rangle$ as the *Kelvin-Noether quantity* ([Holm, Marsden and Ratiu \[1998\]](#)). In application to fluids, \mathcal{C} is the space of loops in the fluid domain and \mathcal{K} is the circulation around this loop, namely

$$\langle \mathcal{K}(c), \mathbf{u} \cdot d\mathbf{x} \rangle := \oint_c \mathbf{u} \cdot d\mathbf{x}.$$

The Kelvin-Noether theorem for Casimir dissipative LP equations is formulated as follows.

Proposition 2.4. Fix $c_0 \in \mathcal{C}$ and consider a solution $\mu(t)$ of the Casimir dissipative LP equation (2.2). Let $g(t) \in G$ be the curve determined by the equation $\frac{\delta h}{\delta \mu} = \dot{g}g^{-1}$, $g(0) = e$. Then the time derivative of the Kelvin-Noether quantity $\langle \mathcal{K}(g(t)c_0), \mu(t) \rangle$ associated to this solution is

$$\frac{d}{dt} \langle \mathcal{K}(g(t)c_0), \mu(t) \rangle = \theta \left\langle \mathcal{K}(g(t)c_0), \text{ad}_{\frac{\delta h}{\delta \mu}}^* \left[\frac{\delta C}{\delta \mu}, \frac{\delta h}{\delta \mu} \right]^b \right\rangle.$$

Proof. The proof is a direct calculation. We have

$$\begin{aligned} \frac{d}{dt} \langle \mathcal{K}(g(t)c_0), \mu(t) \rangle &= \frac{d}{dt} \langle \mathcal{K}(c_0), \text{Ad}_{g(t)}^* \mu(t) \rangle = \left\langle \mathcal{K}(c_0), \text{Ad}_{g(t)}^* (\partial_t \mu + \text{ad}_{\frac{\delta h}{\delta \mu}}^* \mu) \right\rangle \\ &= \theta \left\langle \mathcal{K}(g(t)c_0), \text{ad}_{\frac{\delta h}{\delta \mu}}^* \left[\frac{\delta C}{\delta \mu}, \frac{\delta h}{\delta \mu} \right]^b \right\rangle, \end{aligned}$$

where, at the first equality we used the G -equivariance of \mathcal{K} and at the second equality we used the formula $\frac{d}{dt} \text{Ad}_{g(t)}^* \mu(t) = \text{Ad}_{g(t)}^* (\partial_t \mu(t) + \text{ad}_{\dot{g}(t)g(t)^{-1}}^* \mu(t))$, see, e.g., [Marsden and Ratiu \[1994\]](#). \square

Note that $g(t) \in G$ is the motion in Lagrangian coordinates associated to the evolution of the momentum $\mu(t) \in \mathfrak{g}^*$ in Eulerian coordinates. The θ term is an extra source of circulation with a double commutator. This term is absent in the ordinary Lie-Poisson case (i.e. $\theta = 0$) and therefore in this case the Kelvin-Noether quantity $\langle \mathcal{K}(g(t)c_0), \mu(t) \rangle$ is conserved along solutions.

Remark 2.5 (Three dimensional ideal flows). The Casimir dissipation approach for 3D flows developed in §2.2 follows from the present abstract formulation by choosing the Lie algebra $\mathfrak{g} = \mathfrak{X}_{div}(\mathcal{D})$ of divergence free vector fields on \mathcal{D} , the helicity Casimir C , and the positive symmetric bilinear form $\gamma(\mathbf{u}, \mathbf{v}) = \int_{\mathcal{D}} \mathbf{u} \cdot \Lambda \mathbf{v} d^3x$, where $\Lambda = \text{curl}^{-1} L \text{curl}^{-1}$ with L an arbitrary self-adjoint positive operator. Using $\mathbf{v}^\flat = \Lambda \mathbf{v} \cdot d\mathbf{x}$, it is readily checked that with these choices, equations (2.2) and (2.9) yield (1.10) and (1.15). Note that Proposition 2.4 applied to this case yields the circulation theorem

$$\frac{d}{dt} \int_{c(\mathbf{u})} \mathbf{u} \cdot d\mathbf{x} = \theta \int_{c(\mathbf{u})} \mathcal{L}_{\mathbf{u}} (\Lambda [\mathbf{u}, \boldsymbol{\omega}] \cdot d\mathbf{x}) = -\theta \int_{c(\mathbf{u})} \mathbf{u} \times L(\boldsymbol{\omega} \times \mathbf{u}) \cdot d\mathbf{x},$$

which consistently recovers our previous expression for Kelvin's circulation theorem with Casimir dissipation obtained (1.20) by a direct calculation.

Remark 2.6 (Casimir dissipation VS double bracket dissipation). The Casimir dissipative LP setting considered here is essentially different from the double bracket dissipation setting. Indeed, double bracket equations dissipate energy while they preserve the Casimirs. This is exactly the opposite with what happens in the present setting. There are however apparent similarities in the Lie algebraic formulations. Indeed, on general Lie algebras the double bracket dissipation equations can be written as

$$\frac{df(\mu)}{dt} = \{f, h\}_+(\mu) - \theta \gamma^* \left(\text{ad}_{\frac{\delta k}{\delta \mu}}^* \mu, \text{ad}_{\frac{\delta f}{\delta \mu}}^* \mu \right), \quad (2.11)$$

(compare with equation (2.9)) where γ is a inner product on \mathfrak{g} , γ^* is the inner product induced on \mathfrak{g}^* , and $k : \mathfrak{g}^* \rightarrow \mathbb{R}$ is a given function. One readily checks that Casimir are preserved while, in the special case $k = h$, the energy dissipates. In that case, the equation of motion arising from the double bracket in (2.11) is given by

$$\partial_t \mu + \text{ad}_{\frac{\delta h}{\delta \mu}}^* \mu = \theta \text{ad}_{\left(\text{ad}_{\frac{\delta h}{\delta \mu}}^* \mu\right)^\sharp}^* \mu, \quad (2.12)$$

see Bloch et al. [1996]; Holm, Putkaradze and Tronci [2008], where $\sharp : \mathfrak{g}^* \rightarrow \mathfrak{g}$ is the sharp operator associated to γ . Formula (2.12) for double bracket dissipation may be compared with the corresponding equation for Casimir dissipation in (2.2)

$$\partial_t \mu + \text{ad}_{\frac{\delta h}{\delta \mu}}^* \mu = \theta \text{ad}_{\frac{\delta h}{\delta \mu}}^* \left[\frac{\delta C}{\delta \mu}, \frac{\delta h}{\delta \mu} \right]^\flat, \quad (2.13)$$

and also with the corresponding equation for “anticipated motion” in (2.7)

$$\partial_t \mu + \text{ad}_{\frac{\delta h}{\delta \mu}}^* \mu = \theta \text{ad}_{\frac{\delta h}{\delta \mu}}^* \left(\text{ad}_{\frac{\delta h}{\delta \mu}}^* \mu \right). \quad (2.14)$$

The last three of these formulas all coincide with (1.9) for the case of 2D incompressible flow.

2.2 Lagrange-d'Alembert variational principle.

We now explain how the Casimir dissipative LP equations can be obtained via a variational principle. Consider the Lagrangian $\ell : \mathfrak{g} \rightarrow \mathbb{R}$ related to h via the Legendre transform, that is, we have

$$h(\mu) = \langle \mu, \xi \rangle - \ell(\xi), \quad \mu := \frac{\delta \ell}{\delta \xi},$$

where we assumed that the second relation yields a bijective correspondence between ξ and μ . In terms of ℓ , equation (2.2) for Casimir dissipation reads

$$\partial_t \mu + \text{ad}_\xi^* \mu = \theta \text{ad}_\xi^* \left[\frac{\delta C}{\delta \mu}, \xi \right]^\flat, \quad \mu := \frac{\delta \ell}{\delta \xi}. \quad (2.15)$$

When $\theta = 0$ we recover the *Euler-Poincaré equations*

$$\partial_t \frac{\delta \ell}{\delta \xi} + \text{ad}_\xi^* \frac{\delta \ell}{\delta \xi} = 0$$

and it is well-known that these equations can be obtained via the *constrained variational principle* Arnold and Khesin [1998]; Holm, Marsden and Ratiu [1998]

$$\delta \int_0^T \ell(\xi) dt = 0, \quad \text{for variations} \quad \delta \xi = \partial_t \zeta - [\xi, \zeta],$$

where $\zeta \in \mathfrak{g}$ is an arbitrary curve vanishing at $t = 0, T$.

External forces $f(\xi) \in \mathfrak{g}^*$ can be included in the Euler-Poincaré equations by using the *reduced Lagrange-d'Alembert principle* Bloch [2004]

$$\delta \left[\int_0^T \ell(\xi) dt \right] + \int_0^T \langle f(\xi), \zeta \rangle dt = 0, \quad \text{for variations} \quad \delta \xi = \partial_t \zeta - [\xi, \zeta], \quad (2.16)$$

where $\zeta \in \mathfrak{g}$ is an arbitrary curve vanishing at $t = 0, T$. Substituting into (2.16) the formula for the force appearing in the Casimir dissipation equations (2.15) recovers equations (2.15), now rederived from the Lagrange-d'Alembert variational principle

$$\delta \left[\int_0^T \ell(\xi) dt \right] + \theta \int_0^T \gamma \left(\left[\frac{\delta C}{\delta \mu}, \xi \right], [\xi, \zeta] \right) dt = 0, \quad \text{for variations} \quad \delta \xi = \partial_t \zeta - [\xi, \zeta].$$

Thus, in the Lagrange-d'Alembert formulation, the modification of the motion equation to impose selective decay is seen as an energy-conserving constraint force.

For example, for three dimensional ideal flows, equations (1.15) can be obtained from the variational principle

$$\delta \int_0^T \int_{\mathcal{D}} \frac{1}{2} |\mathbf{u}|^2 d^3x dt + \theta \int_0^T \int_{\mathcal{D}} ([\boldsymbol{\omega}, \mathbf{u}] \cdot \Lambda[\mathbf{u}, \mathbf{v}]) d^3x dt = 0, \quad \text{for variations} \quad \delta \mathbf{u} = \partial_t \mathbf{v} + [\mathbf{u}, \mathbf{v}].$$

The second term on the left hand side can be also written as

$$\theta \int_0^T \int_{\mathcal{D}} (\boldsymbol{\omega} \times \mathbf{u}) \cdot L(\mathbf{u} \times \mathbf{v}) d^3x dt.$$

This is the time integral of the rate of virtual work done by the force for a virtual motion $\mathbf{v} = \delta \varphi \circ \varphi^{-1}$ of the fluid configuration $\varphi : \mathcal{D} \rightarrow \mathcal{D}$.

3 Semidirect product examples

The Hamiltonian structure of fluids that possess advected quantities such as heat, mass, buoyancy, magnetic field, etc., can be understood by using Lie-Poisson brackets for semidirect-product actions of Lie groups on vector spaces.

In this setting, besides the Lie group configuration space G , one needs to include a vector space V on which G acts linearly. Its dual vector space V^* contains the advected quantities. From this, one considers the semidirect product $G \ltimes V$ with Lie algebra $\mathfrak{g} \ltimes V$, and the Hamiltonian structure is given by the Lie-Poisson bracket (2.1), written on $(\mathfrak{g} \ltimes V)^*$ instead of \mathfrak{g}^* . We refer to [Holm, Marsden and Ratiu \[1998\]](#) for a detailed treatment. Given a Hamiltonian function $h = h(\mu, a)$ on $(\mathfrak{g} \ltimes V)^*$ one thus obtains the Lie-Poisson equations

$$\partial_t(\mu, a) + \text{ad}^*_{\left(\frac{\delta h}{\delta \mu}, \frac{\delta h}{\delta a}\right)} \mu = 0,$$

for $\mu(t) \in \mathfrak{g}^*$ and $a(t) \in V^*$. More explicitly, making use of the expression of the ad^* -operator in the semidirect product case, these equations read

$$\partial_t \mu + \text{ad}^*_{\frac{\delta h}{\delta \mu}} \mu + \frac{\delta h}{\delta a} \diamond a = 0, \quad \partial_t a + a \frac{\delta h}{\delta \mu} = 0, \quad (3.1)$$

where the operator $\diamond : V \times V^* \rightarrow \mathfrak{g}^*$ is defined by $\langle v \diamond a, \xi \rangle = -\langle a\xi, v \rangle$, and $a\xi \in V^*$ denotes the Lie algebra action of $\xi \in \mathfrak{g}$ on $a \in V^*$.

We can easily extend the Casimir dissipation approach of §2 to the semidirect product case. Fixing a Casimir function $C = C(\mu, a)$ for the Lie-Poisson bracket on the semidirect product and a (possibly (μ, a) -dependent) positive symmetric bilinear form $\gamma_{(\mu, a)} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$, we extend the semidirect-product Lie-Poisson system (3.1) naturally to allow for Casimir dissipation by setting

$$\partial_t \mu + \text{ad}^*_{\frac{\delta h}{\delta \mu}} \mu + \frac{\delta h}{\delta a} \diamond a = \theta \text{ad}^*_{\frac{\delta C}{\delta \mu}, \frac{\delta h}{\delta \mu}} \flat, \quad \partial_t a + a \frac{\delta h}{\delta \mu} = 0, \quad (3.2)$$

where $\flat : \mathfrak{g} \rightarrow \mathfrak{g}^*$ is the flat operator associated to γ . In Lie-Poisson form, this equation reads

$$\frac{df(\mu, a)}{dt} = \{f, h\}_+(\mu, a) - \theta \gamma \left(\left[\frac{\delta f}{\delta \mu}, \frac{\delta h}{\delta \mu} \right], \left[\frac{\delta C}{\delta \mu}, \frac{\delta h}{\delta \mu} \right] \right),$$

which shows that the energy is conserved while the Casimir dissipates as in (2.10).

To formulate the Kelvin-Noether theorem, we consider a G -equivariant map $\mathcal{K} : \mathcal{C} \times V^* \rightarrow \mathfrak{g}^{**}$, i.e. $\langle \mathcal{K}(gc, ag^{-1}), \text{Ad}^*_{g^{-1}} \nu \rangle = \langle \mathcal{K}(c, a), \nu \rangle, \forall g \in G$. Given $c_0 \in \mathcal{C}$ and a solution $\mu(t), a(t)$ of (3.2), the associated Kelvin-Noether quantity reads $\langle \mathcal{K}(g(t)c_0, a(t)), \mu(t) \rangle$ and similar computations as in Proposition 2.4 yield the Kelvin-Noether theorem

$$\frac{d}{dt} \langle \mathcal{K}(g(t)c_0, a(t)), \mu(t) \rangle = \left\langle \mathcal{K}(g(t)c_0, a(t)), \theta \text{ad}^*_{\frac{\delta C}{\delta \mu}, \frac{\delta h}{\delta \mu}} \flat - \frac{\delta h}{\delta a} \diamond a \right\rangle. \quad (3.3)$$

Remark 3.1 (Natural generalization). Note that the Casimir dissipative LP equation (3.2) is obtained from the Lie-Poisson equations on the semidirect product by modifying the momentum

$\mu \in \mathfrak{g}^*$ only, while keeping the quantity $a \in V^*$ unchanged. From the Lie algebraic point of view, however, the direct generalization of (2.2) to semidirect product Lie groups is

$$\partial_t(\mu, a) + \text{ad}^*_{\left(\frac{\delta h}{\delta \mu}, \frac{\delta h}{\delta a}\right)} \mu = \theta \text{ad}^*_{\left(\frac{\delta h}{\delta \mu}, \frac{\delta h}{\delta a}\right)} \left(\left[\left(\frac{\delta C}{\delta \mu}, \frac{\delta C}{\delta a} \right), \left(\frac{\delta h}{\delta \mu}, \frac{\delta h}{\delta a} \right) \right]^{\flat} \right), \quad (3.4)$$

where the flat operator $\flat : \mathfrak{g} \times V \rightarrow \mathfrak{g}^* \times V^*$ is associated to a positive symmetric bilinear map $\gamma_{(\mu, a)} : (\mathfrak{g} \times V) \times (\mathfrak{g} \times V) \rightarrow \mathbb{R}$. This results in a modification of both μ and a as

$$\tilde{\mu} = \mu - \theta \left[\frac{\delta C}{\delta \mu}, \frac{\delta h}{\delta \mu} \right]^{\flat}, \quad \tilde{a} = a - \theta \left(\frac{\delta C}{\delta a} \frac{\delta h}{\delta \mu} - \frac{\delta h}{\delta a} \frac{\delta C}{\delta \mu} \right)^{\flat}.$$

Of course, as before, the modified semidirect-product Lie-Poisson system (3.4) introduces dissipation of the Casimir C while keeping energy conserved. It recovers (3.2) in the case when the bilinear form γ vanishes on V .

Lagrange-d'Alembert variational principle for semidirect products. As in (2.15), equations (3.2) for Casimir dissipation in the semidirect product case can be expressed from the Lagrangian $\ell : \mathfrak{g} \times V^* \rightarrow \mathbb{R}$ associated to h via the Legendre transformation

$$h(\mu, a) = \langle \mu, \xi \rangle - \ell(\xi, a), \quad \mu := \frac{\delta \ell}{\delta \xi}. \quad (3.5)$$

When $\theta = 0$, the constrained variational principle associated to the equations reads

$$\delta \int_0^T \ell(\xi, a) dt = 0, \quad \text{for variations } \delta \xi = \partial_t \zeta - [\xi, \zeta], \quad \delta a = -a \zeta,$$

where $\zeta \in \mathfrak{g}$ is an arbitrary curve vanishing at $t = 0, T$, see [Holm, Marsden and Ratiu \[1998\]](#). As one may verify directly, when $\theta \neq 0$, the dissipative Casimir force can be included in the Euler-Poincaré equations by considering the Lagrange-d'Alembert variational principle

$$\delta \left[\int_0^T \ell(\xi, a) dt \right] + \theta \int_0^T \gamma \left(\left[\frac{\delta C}{\delta \mu}, \xi \right], [\xi, \zeta] \right) dt = 0, \quad (3.6)$$

for variations $\delta \xi = \partial_t \zeta - [\xi, \zeta]$, $\delta a = -a \zeta$.

3.1 Rotating shallow water (RSW) flows

On the semidirect product of $\text{Diff}(\mathcal{D})$ with functions $\eta \in \mathcal{F}(\mathcal{D})$, where $\text{Diff}(\mathcal{D})$ is the group of diffeomorphisms of a two dimensional domain \mathcal{D} , the Lie-Poisson equations (3.1) become

$$\partial_t \left(\frac{\mathbf{m}}{\eta} \right) + \text{curl} \left(\frac{\mathbf{m}}{\eta} \right) \times \frac{\delta h}{\delta \mathbf{m}} + \nabla \left(\frac{\delta h}{\delta \mathbf{m}} \cdot \frac{\mathbf{m}}{\eta} + \frac{\delta h}{\delta \eta} \right) = 0, \quad \partial_t \eta + \text{div} \left(\eta \frac{\delta h}{\delta \mathbf{m}} \right) = 0. \quad (3.7)$$

The Hamiltonian for the rotating shallow water (RSW) equations is

$$h(\mathbf{m}, \eta) = \int_{\mathcal{D}} \left(\frac{1}{2\eta} |\mathbf{m} - \eta \mathbf{R}|^2 + \frac{1}{2} g(\eta - \bar{D})^2 \right) dx dy, \quad (3.8)$$

where $\text{curl } \mathbf{R} = 2\boldsymbol{\Omega}$ is the Coriolis parameter and $\bar{D}(x, y)$ is the mean depth of the bottom topography in the two dimensional domain with spatial coordinates (x, y) . Indeed, inserting this Hamiltonian into the Lie-Poisson equation (3.7) yields the rotating shallow water equations for the fluid velocity $\mathbf{u} = \mathbf{m}/\eta - \mathbf{R}$ and the total depth η ,

$$\partial_t \mathbf{u} + \text{curl}(\mathbf{u} + \mathbf{R}) \times \mathbf{u} + \nabla \left(\frac{1}{2} |\mathbf{u}|^2 + g(\eta - \bar{D}) \right) = 0, \quad \partial_t \eta + \text{div}(\eta \mathbf{u}) = 0.$$

The second term may be written as $\eta^{-1} \text{curl}(\mathbf{u} + \mathbf{R}) \times \eta \mathbf{u} = q \hat{\mathbf{z}} \times \eta \mathbf{u}$, with the potential vorticity (PV) q given by

$$q = \eta^{-1} \hat{\mathbf{z}} \cdot \text{curl}(\mathbf{m}/\eta) = \eta^{-1} \hat{\mathbf{z}} \cdot \text{curl}(\mathbf{u} + \mathbf{R})$$

As a consequence of the shallow water equations, the potential vorticity q is advected (conserved on fluid parcels), that is

$$\partial_t q + \mathbf{u} \cdot \nabla q = 0.$$

Of course, the conservation of PV is not limited to the shallow water equations. In fact, it holds for the equations derived from this LP structure for any Hamiltonian. This is because the LP structure admits PV in a family of Casimir functions that Poisson commute with every functional of the variables (\mathbf{m}, η) .

One class of Casimir functions is given by

$$C_\Phi(\mathbf{m}, \eta) = \int_{\mathcal{D}} \eta \Phi(q, \eta) dx dy, \quad \text{where } q = \eta^{-1} \hat{\mathbf{z}} \cdot \text{curl}(\mathbf{m}/\eta),$$

for any smooth function Φ . We will compute the Casimir dissipative version of RSW associated to the Casimir function

$$C(\mathbf{m}, \eta) = \frac{1}{2} \int_{\mathcal{D}} \eta q^2 dx dy, \quad (3.9)$$

and the inner product

$$\gamma_\eta(\mathbf{u}, \mathbf{v}) = \int_{\mathcal{D}} \eta (\mathbf{u} \cdot \mathbf{v}) dx dy. \quad (3.10)$$

The flat operator associated to this inner product is thus $\mathbf{v}^\flat = \eta \mathbf{v} \cdot d\mathbf{x}$. From the abstract Lie algebraic formulation (3.2) we obtain

$$\partial_t \mathbf{u} + \text{curl}(\mathbf{u} + \mathbf{R} + \theta[\mathbf{u}, \hat{\mathbf{z}} \times (\nabla q)/\eta]) \times \mathbf{u} + \nabla \left(\frac{1}{2} |\mathbf{u}|^2 + g(\eta - \bar{D}) + \theta \mathbf{u} \cdot [\mathbf{u}, \hat{\mathbf{z}} \times (\nabla q)/\eta] \right) = 0. \quad (3.11)$$

This equation is obtained from the Lie-Poisson equations (3.7), where in the second and third terms the expressions \mathbf{m}/η have been replaced by $\tilde{\mathbf{m}}/\eta$, where

$$\tilde{\mathbf{m}} = \mathbf{m} - \theta \left[\frac{\delta h}{\delta \mathbf{m}}, \frac{\delta C}{\delta \mathbf{m}} \right]^\flat = \mathbf{m} + \theta [\mathbf{u}, \hat{\mathbf{z}} \times (\nabla q)/\eta]^\flat = \mathbf{m} + \theta \eta [\mathbf{u}, \hat{\mathbf{z}} \times (\nabla q)/\eta],$$

as dictated by the Lie-Poisson Casimir dissipation approach in formula (3.2). We have used the formula $\delta C/\delta \mathbf{m} = \text{curl}(q \hat{\mathbf{z}})/\eta = -\hat{\mathbf{z}} \times (\nabla q)/\eta$ for the variational derivative of the Casimir (3.9) and kept in mind that the Lie algebra bracket appearing in (3.2) is minus the Lie bracket of vector fields. The last term of (3.11) is computed as follows

$$\begin{aligned} \frac{\delta h}{\delta \mathbf{m}} \cdot \frac{\tilde{\mathbf{m}}}{\eta} + \frac{\delta h}{\delta \eta} &= \mathbf{u} \cdot (\mathbf{u} + \mathbf{R} + \theta[\mathbf{u}, \hat{\mathbf{z}} \times (\nabla q)/\eta]) - \frac{1}{2} |\mathbf{u}|^2 - \mathbf{u} \cdot \mathbf{R} + g(\eta - \bar{D}) \\ &= \frac{1}{2} |\mathbf{u}|^2 + g(\eta - \bar{D}) + \theta \mathbf{u} \cdot [\mathbf{u}, \hat{\mathbf{z}} \times (\nabla q)/\eta], \end{aligned}$$

where we used the relation $\mathbf{u} = \mathbf{m}/\eta - \mathbf{R}$ between the velocity and momentum in the rotating case.

This model preserves the energy and dissipates the Casimir $\frac{1}{2} \int_{\mathcal{D}} \eta q^2 dx dy$ as

$$\frac{d}{dt} \frac{1}{2} \int_{\mathcal{D}} \eta q^2 dx dy = -\theta \int_{\mathcal{D}} \eta [\hat{\mathbf{z}} \times (\nabla q)/\eta, \mathbf{u}]^2 dx dy.$$

Kelvin circulation theorem. A convenient way to derive the Kelvin circulation theorem (3.3) is to rewrite the Lie-Poisson equations (3.7) using the duality pairing with one-forms. We get

$$\partial_t \left(\frac{\alpha}{\eta} \right) + \mathcal{L}_{\frac{\delta h}{\delta \alpha}} \left(\frac{\alpha}{\eta} \right) + d \frac{\delta h}{\delta \eta} = 0, \quad \partial_t \eta + \operatorname{div} \left(\eta \frac{\delta h}{\delta \alpha} \right) = 0, \quad (3.12)$$

in which α denotes the 1-form $\alpha = \mathbf{m} \cdot d\mathbf{x}$ and $\mathcal{L}_{\delta h/\delta \alpha}$ denotes the Lie derivative with respect to the vector field $\delta h/\delta \alpha$, as defined for 3D earlier in (1.19). The Casimir dissipative LP equation is obtained by replacing α by $\tilde{\alpha} = \tilde{\mathbf{m}} \cdot d\mathbf{x}$ in the second term. Consequently, a direct computation yields

$$\frac{d}{dt} \oint_{c(u)} \frac{\alpha}{\eta} = \theta \oint_{c(u)} \mathcal{L}_{\frac{\delta h}{\delta \alpha}} \left(\frac{1}{\eta} \left[\frac{\delta h}{\delta \alpha}, \frac{\delta C}{\delta \alpha} \right]^b \right) = \theta \oint_{c(u)} \mathcal{L}_{\frac{\delta h}{\delta \alpha}} \left(\left[\frac{\delta h}{\delta \alpha}, \frac{\delta C}{\delta \alpha} \right] \cdot d\mathbf{x} \right),$$

consistently with (3.3), where we recall that the Lie algebraic bracket is minus the Lie bracket of vector fields. For RSW, the previous equation produces the following Kelvin circulation theorem

$$\frac{d}{dt} \oint_{c(u)} (\mathbf{u} + \mathbf{R}) \cdot d\mathbf{x} = \theta \oint_{c(u)} \mathcal{L}_u ([\hat{\mathbf{z}} \times (\nabla q)/\eta, \mathbf{u}] \cdot d\mathbf{x}).$$

Modified PV advection. One way to obtain the modified advection equation for PV, is to take the exterior differential of the first equation in (3.12) (with α replaced by $\tilde{\alpha}$ in the second term) and obtain

$$\partial_t d \left(\frac{\alpha}{\eta} \right) + \mathcal{L}_u d \left(\frac{\alpha}{\eta} \right) = \theta d([\hat{\mathbf{z}} \times (\nabla q)/\eta, \mathbf{u}] \cdot d\mathbf{x}).$$

From this, and from the second equation in (3.12) one obtains

$$\partial_t q + \mathbf{u} \cdot \nabla q = \theta \frac{1}{\eta} \hat{\mathbf{z}} \cdot \operatorname{curl}([\hat{\mathbf{z}} \times (\nabla q)/\eta, \mathbf{u}]),$$

by making use of the formulas

$$\frac{1}{\eta} d \left(\frac{\alpha}{\eta} \right) = q \cdot d\mathbf{S} \quad \text{and} \quad d(\mathbf{m} \cdot d\mathbf{x}) = (\hat{\mathbf{z}} \cdot \operatorname{curl} \mathbf{m}) \cdot d\mathbf{S}$$

for one-forms α and vector fields \mathbf{m} on the two dimensional domain \mathcal{D} .

Lagrange-d'Alembert variational principle. Making use of the Legendre transform (3.5), one obtains the RSW Lagrangian

$$\ell(\mathbf{u}, \eta) = \int_{\mathcal{D}} \left[\eta \left(\frac{1}{2} |\mathbf{u}|^2 + \mathbf{u} \cdot \mathbf{R} \right) - \frac{1}{2} g(\eta - \bar{D}) \right] dx dy.$$

From (3.6) we get the Lagrange-d'Alembert variational principle

$$\delta \int_0^T \ell(\mathbf{u}, \eta) dt + \theta \int_0^T \int_{\mathcal{D}} \eta([\mathbf{u}, \hat{\mathbf{z}} \times (\nabla q)/\eta] \cdot [\mathbf{u}, \mathbf{v}]) dx dy,$$

for variations $\delta \mathbf{u} = \partial_t \mathbf{v} + [\mathbf{u}, \mathbf{v}]$, $\delta \eta = -\operatorname{div}(\eta \mathbf{v})$, where \mathbf{v} is an arbitrary time dependent vector field vanishing at $t = 0, T$, associated to a virtual displacement $\delta \varphi$ of the fluid configuration φ .

3.2 3D rotating Boussinesq flows

On the semidirect product of $\operatorname{SDiff}(\mathbb{R}^3)$ with functions $b \in \mathcal{F}(\mathbb{R}^3)$, where $\operatorname{SDiff}(\mathbb{R}^3)$ is the group of volume preserving diffeomorphisms of \mathbb{R}^3 , the Lie-Poisson equations (3.1) become

$$\partial_t \mathbf{q} + \operatorname{curl} \left(\mathbf{q} \times \operatorname{curl} \frac{\delta h}{\delta \mathbf{q}} \right) - \nabla \frac{\delta h}{\delta b} \times \nabla b = 0, \quad \partial_t b + \operatorname{curl} \frac{\delta h}{\delta \mathbf{q}} \cdot \nabla b = 0. \quad (3.13)$$

For the rotating 3D Boussinesq model, the Hamiltonian is

$$h(\mathbf{q}, b) = \int \left(\frac{1}{2} |\operatorname{curl}^{-1} \mathbf{q} - \mathbf{R}|^2 + bz \right) d^3x,$$

where b is the scalar buoyancy and $\operatorname{curl} \mathbf{R} = 2\Omega$ is the Coriolis parameter. The variational derivatives are

$$\operatorname{curl} \frac{\delta h}{\delta \mathbf{q}} = \operatorname{curl}^{-1} \mathbf{q} - \mathbf{R} = \mathbf{u}, \quad \frac{\delta h}{\delta b} = z,$$

and the vorticity is $\boldsymbol{\omega} = \operatorname{curl} \mathbf{u} = \mathbf{q} - \operatorname{curl} \mathbf{R} = \mathbf{q} - 2\Omega$. In this case, the Lie-Poisson equations (3.13) reduce to the Rotating 3D Boussinesq equations

$$\partial_t \boldsymbol{\omega} + \operatorname{curl} ((\boldsymbol{\omega} + 2\Omega) \times \mathbf{u}) - \mathbf{z} \times \nabla b = 0, \quad \partial_t b + \mathbf{u} \cdot \nabla b = 0,$$

or, in velocity form

$$\partial_t \mathbf{u} + (\operatorname{curl} \mathbf{u} + 2\Omega) \times \mathbf{u} + \mathbf{z} b = -\nabla p,$$

where p is the pressure, found by enforcing incompressibility, $\operatorname{div} \mathbf{u} = 0$.

The Lie-Poisson equations (3.13) admit the Casimir functions

$$C_\Phi(\mathbf{q}, b) = \int_{\mathcal{D}} \Phi(q, b) d^3x, \quad \text{where } q := \nabla b \cdot \mathbf{q} \text{ is the potential vorticity.}$$

We shall now introduce the dissipative Casimir effect in the 3D Rotating Boussinesq system by using the Lie algebraic setting developed above. Choosing the *enstrophy* Casimir

$$C(\mathbf{q}, b) = \frac{1}{2} \int_{\mathcal{D}} q^2 d^3x, \quad (3.14)$$

we have

$$\operatorname{curl} \frac{\delta C}{\delta \mathbf{q}} = \operatorname{curl}(q \nabla b), \quad \frac{\delta C}{\delta b} = -\operatorname{div}(q \mathbf{q}).$$

Consequently, choosing the inner product γ associated to a given positive self-adjoint differential operator Λ , the modified momentum (2.3) is

$$\tilde{\mathbf{q}} = \mathbf{q} - \theta \operatorname{curl} \Lambda [\mathbf{u}, \operatorname{curl}(q \nabla b)] = \mathbf{q} - \theta L(\operatorname{curl}(q \nabla b) \times \mathbf{u}),$$

where the differential operator L verifies $\Lambda = \text{curl}^{-1} L \text{curl}^{-1}$. The dissipative Casimir Lie-Poisson equations $\partial_t \mathbf{q} + \text{curl} \left(\tilde{\mathbf{q}} \times \text{curl} \frac{\delta h}{\delta \mathbf{q}} \right) - \nabla \frac{\delta h}{\delta b} \times \nabla b = 0$, yields, in velocity form, the system

$$\partial_t \mathbf{u} + \text{curl} (\mathbf{u} + \mathbf{R} - \theta \Lambda [\mathbf{u}, \text{curl}(q \nabla b)]) \times \mathbf{u} + b \hat{\mathbf{z}} = -\nabla p, \quad \partial_t b + \mathbf{u} \cdot \nabla b = 0.$$

The first equation can be rewritten, in terms of L as

$$\partial_t \mathbf{u} + (\text{curl} (\mathbf{u} + \mathbf{R}) - \theta L(\text{curl}(q \nabla b) \times \mathbf{u})) \times \mathbf{u} + b \hat{\mathbf{z}} = -\nabla p. \quad (3.15)$$

This system preserves the energy $h(\mathbf{q}, b)$ of the 3D rotating Boussinesq fluid and dissipates the enstrophy as

$$\frac{d}{dt} \frac{1}{2} \int_{\mathcal{D}} q^2 d^3 x = -\theta \left\| [\text{curl}(q \nabla b), \mathbf{u}] \right\|_{\Lambda}^2 = -\theta \left\| \text{curl}(q \nabla b) \times \mathbf{u} \right\|_L^2, \quad (3.16)$$

where we have used the identity (1.24).

Remark 3.2. In the Casimir dissipation approach for the 3D Boussinesq equations, the vector $\text{curl}(q \nabla b)$ in equations (3.15) and (3.16) now plays the role that the vorticity $\boldsymbol{\omega}$ played for ideal incompressible 3D in equations (1.16) and (1.22).

Kelvin circulation theorem. A convenient way to derive the Kelvin circulation theorem (3.3) is to rewrite the modification of equations (3.13) using duality pairing with one-forms. In this case, we have

$$\partial_t \alpha + \mathcal{L}_{\frac{\delta h}{\delta \alpha}} \tilde{\alpha} - \frac{\delta h}{\delta b} db = -dp,$$

in which α denotes the 1-form $\alpha = \boldsymbol{\alpha} \cdot d\mathbf{x}$ or, more explicitly,

$$\partial_t \alpha + \mathcal{L}_{\frac{\delta h}{\delta \alpha}} \alpha = \theta \mathcal{L}_{\frac{\delta h}{\delta \alpha}} \left[\frac{\delta h}{\delta \alpha}, \frac{\delta C}{\delta \alpha} \right]^b + \frac{\delta h}{\delta b} db - dp.$$

Consequently, a direct computation yields

$$\frac{d}{dt} \oint_{c(u)} \alpha = \theta \oint_{c(u)} \mathcal{L}_{\frac{\delta h}{\delta \alpha}} \left[\frac{\delta h}{\delta \alpha}, \frac{\delta C}{\delta \alpha} \right]^b - \oint_{c(u)} b d \frac{\delta h}{\delta b},$$

consistently with (3.3), where we again recall that the Lie algebraic bracket is minus the Lie bracket of vector fields. For rotating 3D Boussinesq, upon using the enstrophy as a Casimir function and employing the differential operator Λ , one gets

$$\begin{aligned} \frac{d}{dt} \oint_{c(u)} (\mathbf{u} + \mathbf{R}) \cdot d\mathbf{x} &= \theta \oint_{c(u)} \mathcal{L}_{\mathbf{u}} (\Lambda [\mathbf{u}, \text{curl}(q \nabla b)] \cdot d\mathbf{x}) - \oint_{c(u)} b dz \\ &= -\theta \oint_{c(u)} \mathbf{u} \times L(\text{curl}(q \nabla b) \times \mathbf{u}) \cdot d\mathbf{x} - \oint_{c(u)} b dz, \end{aligned}$$

where we have used the expression for the Hamiltonian in terms of $\boldsymbol{\alpha}$ and b , that is,

$$h(\alpha, b) = \int_{\mathcal{D}} \left(\frac{1}{2} |\boldsymbol{\alpha} - \mathbf{R}|^2 + bz \right) d^3 x, \quad \frac{\delta h}{\delta \alpha} = \mathbf{u}, \quad \text{with} \quad \boldsymbol{\alpha} = \mathbf{u} + \mathbf{R}.$$

Setting $\theta = 0$ recovers the usual Kelvin circulation theorem for rotating Boussinesq flows.

Lagrange-d'Alembert variational principle for Boussinesq flows. Proceeding exactly as in the preceding example for RSW and using the abstract formulation (3.6), one obtains the Lagrange-d'Alembert variational principle for Boussinesq flows,

$$\delta \int_0^T \ell(\mathbf{u}, b) dt + \theta \int_0^T \int_{\mathcal{D}} (\text{curl}(q\nabla b) \times \mathbf{u}) \cdot L(\mathbf{u} \times \mathbf{v}) dx dy = 0,$$

for variations $\delta \mathbf{u} = \partial_t \mathbf{v} + [\mathbf{u}, \mathbf{v}]$, $\delta b = -\mathbf{v} \cdot \nabla b$. Here \mathbf{v} is an arbitrary time dependent vector field vanishing at $t = 0, T$, that is associated to a virtual displacement $\delta \varphi$ of the fluid configuration $\varphi \in \text{SDiff}(\mathbb{R}^3)$. Also, ℓ is the Lagrangian of the 3D rotating Boussinesq fluid, given by

$$\ell(\mathbf{u}, b) = \int_{\mathcal{D}} \left(\frac{1}{2} |\mathbf{u}|^2 + \mathbf{u} \cdot \mathbf{R} - bz \right) d^3x.$$

Remarkably, in the Casimir dissipation approach for 3D Boussinesq flows, the vector $\text{curl}(q\nabla b)$ now plays the same role as the vorticity $\boldsymbol{\omega}$ had played in the case of 3D ideal fluids. In particular, the vector field $\boldsymbol{\omega} \times \mathbf{u}$ that appeared in the dissipative term for ideal fluids in the 3D Boussinesq case is replaced by the vector $\text{curl}(q\nabla b) \times \mathbf{u}$.

4 Conclusion

This paper has introduced a theory of selective decay by Casimir dissipation that applies widely in fluids and suggests a connection between selective decay and the design of numerical methods, since it recovers some of the previous methods, such as the anticipated vorticity method.

Interestingly, while the entropy C_2 in (1.5) is the appropriate Casimir for selective decay in 2D incompressible flow, the appropriate Casimir for that role in 3D incompressible flow is the *square* of the helicity C in (1.13).

Perhaps one reason for these associations with different Casimirs in 2D and 3D can be suggested by comparing the Lie-Poisson brackets in terms of the vorticity in 2D and 3D. In particular, these brackets may be written in terms of the Casimirs, the enstrophy C_2 in 2D and the helicity C in 3D. In fact, the Lie-Poisson brackets in 2D and 3D may both be written in terms of the corresponding Casimirs in the same triple product form Nambu [1973]. Namely,

$$\{f, h\}_2(\omega) = \int_{\mathcal{D}} \frac{\delta C_2}{\delta \omega} \left[\frac{\delta f}{\delta \omega}, \frac{\delta h}{\delta \omega} \right] dx dy \quad \text{and} \quad \{f, h\}_3(\boldsymbol{\omega}) = \int_{\mathcal{D}} \text{curl} \frac{\delta C}{\delta \boldsymbol{\omega}} \cdot \text{curl} \frac{\delta f}{\delta \boldsymbol{\omega}} \times \text{curl} \frac{\delta h}{\delta \boldsymbol{\omega}} d^3x.$$

These formulas suggest the fundamental roles of the two different Casimirs, the enstrophy C_2 in 2D for which $\delta C_2 / \delta \omega = \omega$ and the helicity C in 3D for which $\delta C / \delta \boldsymbol{\omega} = \mathbf{u}$. One could imagine using other constants of motion instead of the Casimirs, but these would result by Noether's theorem from particular symmetries and their use would be restricted to those subcases.

The loss of both Kelvin's theorem and the Casimir conservation laws when modifying the equations to impose selective decay is consistent with the effects of viscosity, but of course the preservation of energy is not. This, however, is the essence of selective decay, which is seen in turbulence on times scales less than the time scale for viscous decay of energy.

The phenomenon of selective decay is thought to arise from a multiscale interaction between disparate large and small scales in which the large scales are regarded as a type of coarse graining, or

perhaps the envelope, of the small scale motions, [Mininni, Pouquet and Sullivan \[2008\]](#). This type of model is consistent with other recently developed multiscale turbulence models, particularly that in [Holm and Tronci \[2012\]](#), in which the large scale motion is regarded as a *Lagrange coordinate* for the small scale motion.

Thus, a phenomenological point of view exists in which the effect of selective decay of Casimirs that depend on gradients of the velocity may be regarded usefully as providing a type of coarse-graining that is similar to the effect of a turbulent viscosity, but still conserves energy. In this regard, see also [Vallis and Hua \[1988\]](#); [Levy, Dubrulle and Chavanis \[2006\]](#).

This interpretation of selective decay arising from coarse graining may also suggest a connection between selective decay and the design of numerical methods. In particular, one may wish to design numerical methods whose solutions imitate selective decay without adding too much linear viscosity. And from our discussion here it seems that the method of anticipated vorticity in [Sadourny and Basdevant \[1981, 1985\]](#) may be one of those numerical methods.

The theoretical approach presented here may lead to classes of other numerical methods based on selective Casimir decay that would generalize and extend the applicability of the method of anticipated vorticity to other areas of fluid dynamics. A recent discussion of numerical methods based on anticipated vorticity for 2D flows is given in [Chen, Gunzburger, and Ringler \[2011a\]](#). A “scale-aware” version of the anticipated vorticity method in 3D should also be readily obtainable, by allowing the value of its time-scale parameter θ to depend on local mean properties of the solution. However, this is beyond the scope of the present work.

Future applications of our approach here would naturally follow recent work in ocean dynamics that interprets selective decay as a mechanism for parameterizing the interactions between disparate scales, for example, between large coherent oceanic flows and the much smaller eddies directly, as done in [Marshall and Adcroft \[2010\]](#), instead of relying on the slower, indirect effects of viscosity. The Lie-Poisson structures in the selective decay models presented here suggest a framework in which numerical schemes for their computational simulations may be designed. This framework may turn out to be similar to the ideas in [Salmon \[2005\]](#) about using the triple product Nambu form to design simulation algorithms for certain grid structures used in oceanography. In addition, Voronoi mesh methods that were developed for anticipated vorticity dynamics may turn out to be naturally adaptable to the selective decay models presented here. See, e.g., [Chen, Gunzburger, and Ringler \[2011a,b\]](#). However, applications of these ideas for numerical simulation algorithms within the Casimir dissipation modelling framework are beyond our present scope and will be pursued elsewhere.

If selective Casimir decay or the anticipated vorticity method does mimic the effect of linear Navier-Stokes viscosity, but at a convenient time scale for numerical simulations, then one must be cautious about adding in other types of viscosity, such as the artificial viscosity used in Large Eddy Simulations to mimic the effects of turbulence. The introduction of other types of turbulent viscosities such as Smagorinsky tensor diffusivity in *addition* to anticipated vorticity for selective decay would be seen as double counting. In contrast, the introduction of nondissipative regularization through nonlinear dispersion as a modification of the kinetic energy Hamiltonian, as accomplished in the Navier-Stokes alpha (NS-alpha) turbulence model [Foias, Holm and Titi \[2001\]](#), is complementary to selective decay and cannot be regarded as double counting of dissipation. Moreover, the original derivation of the NS-alpha turbulence model by the method of Lagrangian averaging makes it clear that the model is already “scale-aware” because the length scale alpha in that derivation is the correlation length for Lagrangian fluctuations of the fluid parcel paths.

Analytical issues such as whether the selective decay equations admit existence of unique globally strong solutions are not addressed here. However, in combining the NS-alpha model with the present selective decay approach, one would expect existence and uniqueness to persist, because this property is already possessed by the NS-alpha turbulence model.

The Casimir dissipation approach for other fluid models such as magnetohydrodynamics (MHD), compressible fluids, two-layer quasi-geostrophic models parameterizing baroclinic instability, Maxwell-fluid and Maxwell-Vlasov plasmas, spin chains, solitons, complex fluids, turbulence models, etc., will be investigated elsewhere.

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